

# PENROSE-TYPE INEQUALITIES WITH A EUCLIDEAN BACKGROUND

JEFFREY L. JAUREGUI

**ABSTRACT.** We recall the Riemannian Penrose inequality, a geometric statement that restricts the asymptotics of manifolds of nonnegative scalar curvature that contain compact minimal hypersurfaces. The inequality is known for manifolds of dimension up to seven. One source of motivation for the present work is to prove a weakened inequality in all dimensions for conformally flat manifolds. Along the way, we establish new inequalities, including some that apply to manifolds that are merely conformal to a particular type of scalar flat manifold (not necessarily conformally flat).

We also apply the techniques to asymptotically flat manifolds containing zero area singularities, objects that generalize the naked singularity of the negative mass Schwarzschild metric. In particular we derive a lower bound for the ADM mass of a conformally flat, asymptotically flat manifold containing any number of zero area singularities.

## 1. INTRODUCTION

The positive mass theorem (PMT) is a beautiful result on the geometry of manifolds of nonnegative scalar curvature. For instance, it implies a scalar curvature rigidity statement for Euclidean space, and is a crucial ingredient in the solution of the Yamabe problem [18]. The PMT was first proved by Schoen and Yau in dimensions three through seven [19, 20], and later by Witten for spin manifolds [24] in dimensions three or greater. In section 2 we recall the relevant definitions.

**Theorem 1** (Positive mass theorem). *Let  $(M, g)$  be a complete, asymptotically flat Riemannian  $n$ -manifold without boundary, with either  $3 \leq n \leq 7$  or  $M$  a spin manifold. Suppose  $(M, g)$  has nonnegative scalar curvature, and ADM mass  $m$ . Then  $m \geq 0$ , and  $m = 0$  if and only if  $(M, g)$  is isometric to  $\mathbb{R}^n$  with the flat metric.*

The well-known physical interpretation of the PMT is that, in an  $(n + 1)$ -dimensional Lorentzian spacetime obeying the dominant energy condition, any totally geodesic “space-like slice” (Riemannian submanifold of dimension  $n$ ) has nonnegative total mass  $m_{ADM}$  (see [5, 20] for instance).

An important generalization of the PMT is the Riemannian Penrose inequality (RPI), proved as stated below by Bray [5] (for dimension  $n = 3$ ) and later by Bray and Lee [9] (for  $3 \leq n \leq 7$ ). Huisken and Ilmanen gave a proof for  $n = 3$ , with  $|\Sigma|_g$  replaced by the area of the largest connected component of  $\partial M$  [12]. The Bray and Bray–Lee proofs rely on the PMT, while the Huisken–Ilmanen approach provides an independent proof of the PMT for  $n = 3$ .

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**Theorem 2** (Riemannian Penrose inequality). *Let  $(M^n, g)$  be an asymptotically flat manifold of dimension  $3 \leq n \leq 7$  with nonnegative scalar curvature. Suppose the boundary  $\Sigma = \partial M$  has area  $|\Sigma|_g$ , zero mean curvature, and is area-outer-minimizing (see below). Then*

$$m_{ADM}(g) \geq \frac{1}{2} \left( \frac{|\Sigma|_g}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}. \quad (1)$$

We say  $\Sigma$  is *area-outer-minimizing* if every surface enclosing  $\Sigma$  has area at least as large as  $\Sigma$ . The typical physical interpretation of the RPI is as follows (see [5] for more details): view  $(M, g)$  as a totally geodesic spacelike slice of a spacetime obeying the dominant energy condition. Each component of the minimal boundary  $\partial M$  is the apparent horizon of a black hole. The number  $\frac{1}{2} \left( \frac{|\Sigma|_g}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}$  represents the total “mass” of the collection black holes. Thus, the RPI states that the total mass of  $(M, g)$  is at least the mass contributed by the black holes.

We are motivated primarily by two questions:

- Is there evidence to support the RPI in high dimensions?
- Is there a more elementary proof of the RPI for metrics with special properties, such as conformal flatness?

To put these questions in context, first note that a proof of the PMT in all dimensions would not automatically imply the RPI in all dimensions via the Bray–Lee approach. Indeed, there are additional complications in higher dimensions pertaining to the singularities of area-minimizing hypersurfaces. Second, for conformally-flat metrics, the positive mass theorem has a very elementary proof in all dimensions, essentially following from Stokes’ theorem (see [4] for instance). We also refer to reader to the work of Bray and Iga, who proved a version of the RPI for conformally-flat metrics in dimension three, with suboptimal constant [6]. We also mention the recent work of Schwartz and Schwartz–Freire on a “volumetric” Penrose inequality [21, 22] for conformally flat manifolds, and that of Lam, who gave a complete picture of the PMT and RPI in all dimensions for graphical submanifolds of  $\mathbb{R}^{n+1}$  [13, 14].

The present paper is organized as follows. In the next section, we recall the definitions of asymptotic flatness and ADM mass. Section 3 states and proves some Penrose-like inequalities for metrics that are conformal to a special background metric. In Theorem 5, the background manifold is Euclidean space minus a union of domains, and the area on the right-hand side is replaced with the Euclidean area of the boundary. Theorem 7 is a generalization, allowing for the background to be any asymptotically flat, zero scalar curvature manifold that agrees to first order with the Euclidean metric on its compact boundary. Finally, Theorem 8 is a further generalization that allows for possibly negative scalar curvature.

In section 4, we recall the notion of a *zero area singularity*, or ZAS, which can loosely be viewed as a negative-mass black hole [7, 16]. While of less physical interest than black holes, ZAS are natural objects to study in a geometric context. Theorem 10 is a type of Penrose inequality for ZAS contained in conformally flat manifolds. Such an inequality is unknown in general (without conformal flatness), even in low dimensions.

In section 5, we discuss a conjectured inequality for asymptotically flat manifolds that contain both minimal surfaces and zero area singularities. We apply the techniques of the previous sections to prove a weaker version of the inequality in the conformally flat case, under an additional technical assumption.

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## 2. PRELIMINARIES: ASYMPTOTIC FLATNESS AND ADM MASS

Experts may prefer to skip this brief section.

*Definition 3.* A smooth, connected, Riemannian manifold  $(M, g)$  (possibly with compact boundary) of dimension  $n \geq 3$  is **asymptotically flat** (with one end) if

- (i) there exists a compact subset  $K \subset M$  and a diffeomorphism  $\Phi : M \setminus K \rightarrow \mathbb{R}^n \setminus \overline{B}$  (where  $\overline{B}$  is a closed ball), and
- (ii) in the “asymptotically flat coordinates”  $(x^1, \dots, x^n)$  on  $M \setminus K$  induced by  $\Phi$ , the metric  $g$  obeys the decay conditions:

$$\begin{aligned} |g_{ij} - \delta_{ij}| &\leq \frac{c}{|x|^p}, & |\partial_k g_{ij}| &\leq \frac{c}{|x|^{p+1}}, \\ |\partial_k \partial_l g_{ij}| &\leq \frac{c}{|x|^{p+2}}, & |R| &\leq \frac{c}{|x|^q}, \end{aligned}$$

for  $|x| = \sqrt{(x^1)^2 + \dots + (x^n)^2}$  sufficiently large and all  $i, j, k, l = 1, \dots, n$ , where  $c > 0$ ,  $p > \frac{n-2}{2}$ , and  $q > n$  are constants,  $\delta_{ij}$  is the Kronecker delta,  $\partial_k = \frac{\partial}{\partial x^k}$ , and  $R$  is the scalar curvature of  $g$ .

Suppose  $g = u^{\frac{4}{n-2}} \bar{g}$  for a smooth function  $u$ , where  $g$  and  $\bar{g}$  are asymptotically flat metrics. This implies in particular that  $u \rightarrow c$  uniformly at infinity (for some constant  $c > 0$ ),  $|\bar{\nabla} u|_{\bar{g}} = O(r^{-(p+1)})$ , and  $\bar{\Delta} u = O(r^{-q})$ , where  $\bar{\nabla}$  and  $\bar{\Delta}$  are the gradient and Laplacian with respect to  $\bar{g}$ . For later reference, we note that  $|\bar{\nabla} u|_{\bar{g}}^2$  and  $\bar{\Delta} u$  are integrable functions on  $(M, \bar{g})$ .

Next, we recall the definition of the ADM mass [1], a number associated to any asymptotically flat manifold, which provides a measure of the rate at which the metric becomes flat near infinity. Bartnik proved that the ADM mass is a geometric invariant [2].

*Definition 4.* The **ADM mass** of an asymptotically flat manifold  $(M, g)$  is the number

$$m_{ADM}(g) = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r \rightarrow \infty} \sum_{i,j=1}^n \int_{S_r} (\partial_i g_{ij} - \partial_j g_{ii}) \frac{x^j}{r} dA$$

where  $(x^i)$  are asymptotically flat coordinates,  $S_r$  is the coordinate sphere  $\{|x| = r\}$ , and  $\omega_{n-1}$  is the area of the unit sphere in  $\mathbb{R}^n$ .

## 3. INEQUALITIES FOR BLACK HOLES

Let  $\Omega \subset \mathbb{R}^n$  be a bounded open set with smooth boundary  $\partial\Omega$ . Recall that  $\Omega$  is *mean-convex* if  $\partial\Omega$  has nonnegative mean curvature in the outward direction.  $\Omega$  is *star-shaped* if there exists a point  $x_0 \in \Omega$  such that for each  $x \in \Omega$ , the line segment from  $x$  to  $x_0$  is contained in  $\Omega$ .

Our first result is a Penrose-like inequality in all dimensions for conformally flat, asymptotically flat metrics on the complement of  $\Omega$ .

*Theorem 5. Let  $n \geq 3$ , and suppose  $M = \mathbb{R}^n \setminus \Omega$  for a smooth, mean-convex, bounded open set  $\Omega$ , of which every connected component is star-shaped. Let  $k$  be the number of components of  $\Omega$ . Suppose  $g$  is a Riemannian metric on  $M$  of the form  $u^{\frac{4}{n-2}}\delta$ , where  $\delta$  is the flat metric on  $M$  and  $u$  is a smooth positive function tending to one at infinity. Assume that  $g$  is asymptotically flat, has nonnegative scalar curvature  $R_g$ ,  $\partial M = \partial\Omega$  is minimal in  $(M, g)$ . Then*

$$m_{ADM}(g) \geq \left(\frac{A_1}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} + \dots + \left(\frac{A_k}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} + \frac{1}{2(n-1)\omega_{n-1}} \int_M R_g u^{-1} dV_g, \quad (2)$$

where  $A_i$  is the Euclidean area of the  $i$ th connected component of  $\partial\Omega$ ,  $\omega_{n-1}$  is the Euclidean area of the unit  $(n-1)$ -sphere in  $\mathbb{R}^n$ , and  $dV_g$  is the volume form of  $(M, g)$ .

Note that the class of metrics considered above includes, for instance, the Schwarzschild metric of mass  $m > 0$ : choose  $\Omega$  to be the ball of radius  $\left(\frac{m}{2}\right)^{\frac{1}{n-2}}$  about the origin, and take the following metric on  $M = \mathbb{R}^n \setminus \Omega$ :

$$g_m = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \delta.$$

Before presenting the proof, we state some immediate consequences:

*Corollary 6. Under the hypotheses of Theorem 5, we have the following Penrose-like inequalities*

$$m_{ADM}(g) \geq \left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} \quad (3)$$

$$m_{ADM}(g) \geq \left(\frac{V}{\beta_n}\right)^{\frac{n-2}{n}}, \quad (4)$$

where  $A$  is the Euclidean area of  $\partial\Omega$ ,  $V$  is the Euclidean volume of  $\Omega$ , and  $\beta_n$  is the Euclidean volume of the unit  $n$ -ball in  $\mathbb{R}^n$ .

In the course of the proof, we will see that (2), (3), and (4) are not sharp. In the model case in which  $\Omega$  is the unit  $n$ -ball and  $g$  is a Schwarzschild metric, the estimates are suboptimal by a factor of two.

*Remarks.*

- Inequality (3) deceptively appears to be a factor of two stronger than the Riemannian Penrose inequality (1) – but this is not the case, since  $A$  is the area of  $\partial\Omega$  with

respect to the Euclidean metric, not with respect to  $g$ . However, note that we did not assume  $\Sigma$  is area-outer-minimizing in  $(M, g)$ .

- If  $\Sigma$  is assumed to be area-outer-minimizing in  $(M, g)$ , then inequalities (3) and (4) do follow from the RPI (in dimensions  $3 \leq n \leq 7$ ).
- Inequality (4) is the volumetric Penrose inequality recently proved by Schwartz without the star-convexity assumption [21] (c.f. [22]).

*Proof of Theorem 5.* The scalar curvature  $R_g$  of the conformally flat metric  $g = u^{\frac{4}{n-2}} \delta$  is given by the formula

$$R_g = -\frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} \Delta u, \quad (5)$$

where  $\Delta = \text{div grad}$  is the Euclidean Laplacian. Therefore the hypothesis of nonnegative scalar curvature translates to  $\Delta u \leq 0$ . Integrate  $\Delta u$  by parts (with respect to the Euclidean metric) over the region  $B_r$  in  $M$  bounded by a large coordinate sphere  $S_r$  of radius  $r$ :

$$\int_{B_r} \Delta u dV = \int_{S_r} \partial_\nu(u) dA - \int_{\partial M} \partial_\nu(u) dA,$$

where  $dV$ ,  $dA$  and  $\nu$  are the volume form, area form, and unit normal with respect to the Euclidean metric. In both cases,  $\nu$  is chosen to point toward infinity, and  $\partial_\nu$  is the directional derivative with respect to  $\nu$ . Since  $R_g$  is integrable with respect to  $g$  (from the definition of asymptotic flatness), formula (5) and the fact  $u \rightarrow 1$  at infinity proves that  $\Delta u$  is integrable with respect to  $\delta$ . Thus, we may take  $\lim_{r \rightarrow \infty}$  of the above. Multiplying by an appropriate constant, we have

$$-\frac{2}{(n-2)\omega_{n-1}} \lim_{r \rightarrow \infty} \int_{S_r} \partial_\nu(u) dA = -\frac{2}{(n-2)\omega_{n-1}} \int_{\partial M} \partial_\nu(u) dA - \frac{2}{(n-2)\omega_{n-1}} \int_M \Delta u dV.$$

It is straightforward to check that the left-hand side is the ADM mass  $m$  of  $(M, g)$ . Moreover, formula (5) and the fact that the volume form  $dV_g$  of  $g$  equals  $u^{\frac{2n}{n-2}} dV$  reduces this to:

$$m = -\frac{2}{(n-2)\omega_{n-1}} \int_{\partial M} \partial_\nu(u) dA + \frac{1}{2(n-1)\omega_{n-1}} \int_M R_g u^{-1} dV_g, \quad (6)$$

where  $m = m_{ADM}(g)$ . Using the conformal transformation law for the mean curvature and the hypothesis that  $\partial M$  has zero mean curvature with respect to  $g$ , we have:

$$0 = u^{-\frac{n}{n-2}} \left( H u + \frac{2(n-1)}{n-2} \partial_\nu(u) \right) \quad \text{on } \partial M$$

where  $H$  is the Euclidean mean curvature<sup>1</sup> of  $\partial\Omega$ ; this may be rearranged to

$$-\frac{2}{(n-2)\omega_{n-1}} \partial_\nu(u) = \frac{1}{(n-1)\omega_{n-1}} H u. \quad (7)$$

Combining (6) and (7), we obtain:

$$m - \frac{1}{2(n-1)\omega_{n-1}} \int_M R_g u^{-1} dV_g \geq \frac{1}{(n-1)\omega_{n-1}} \int_{\partial M} H u dA \geq \frac{1}{(n-1)\omega_{n-1}} \int_{\partial M} H dA, \quad (8)$$

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<sup>1</sup>We use the convention that  $H$  is the sum, not the average, of the principal curvatures.

having used the fact  $u \geq 1$  (which follows from the maximum principle, noting that  $u$  is a superharmonic function that is one at infinity and has  $\partial_\nu(u) \leq 0$  on  $\partial M$ ; c.f. Lemma 11 of Schwartz).

We apply the ‘‘Minkowski inequality’’ relating the integral of mean curvature of  $\partial\Omega$  to its boundary area<sup>2</sup>. Let  $\Omega_1, \dots, \Omega_k$  be the connected components of  $\Omega$  with boundaries  $\Sigma_1, \dots, \Sigma_k$ . Since we assume  $\Omega_i$  is mean-convex and star-shaped, a theorem of Guan and Li [11] shows

$$\frac{1}{(n-1)\omega_{n-1}} \int_{\Sigma_i} H dA \geq \left( \frac{A_i}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}, \quad (9)$$

for each  $i = 1, \dots, k$ . Combining this with (8) and noting  $\partial M = \Sigma_1 \dot{\cup} \dots \dot{\cup} \Sigma_k$ , we have

$$m - \frac{1}{2(n-1)\omega_{n-1}} \int_M R_g u^{-1} dV_g \geq \left( \frac{A_1}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} + \dots + \left( \frac{A_k}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}.$$

□

*Remarks.* There is precedent for using the Minkowski inequality to prove Penrose-like inequalities; for instance, see the work of Gibbons on collapsing shells [10] and Lam on the case of graphs over  $\mathbb{R}^n \setminus \Omega$  [13].

*Proof of Corollary 6.* Inequality (3) follows immediately, since the weighted integral of scalar curvature is nonnegative, and

$$\left( \frac{A_1}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} + \dots + \left( \frac{A_k}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} \geq \left( \frac{A_1 + \dots + A_k}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}.$$

To prove (4), simply follow (3) by the classical isoperimetric inequality [15] for  $\Omega$ :

$$\left( \frac{A}{\omega_{n-1}} \right)^{\frac{1}{n-1}} \geq \left( \frac{V}{\beta_n} \right)^{\frac{1}{n}}.$$

□

We conclude this section by remarking that it may be possible in Theorem 5 and Corollary 6 to merely assume that  $\Omega$  is area-outer-minimizing in  $\mathbb{R}^n$  (instead of mean-convex and star-shaped). The missing ingredient is the Minkowski inequality (9) for such domains. Such an inequality has been announced by Huisken (see [11]) and also very recently by Freire and Schwartz [22].

**3.1. Beyond the conformally flat case.** The following theorem is a generalization of Theorem 5 in which a scalar-flat background metric  $\bar{g}$  takes on the role of the flat metric  $\delta$ .

*Theorem 7.* Suppose  $M = \mathbb{R}^n \setminus \Omega$  for a smooth, mean-convex, bounded open set  $\Omega$ , of which every connected component is star-shaped. Let  $\bar{g}$  be any asymptotically flat metric on  $M$  with the following properties:

- (1)  $\bar{g}$  has zero scalar curvature
- (2)  $\bar{g}$  and  $\delta$  agree on  $\partial M$

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<sup>2</sup>Other names for this type of estimate are the Aleksandrov–Fenchel inequality and the isoperimetric inequality for quermassintegrals [17]. Classically, it was proved for convex regions in  $\mathbb{R}^3$  by Minkowski [15].

(3)  $\bar{g}$  and  $\delta$  induce the same mean curvature on  $\partial M$ .

Let  $g = u^{\frac{4}{n-2}}\bar{g}$ , where  $u$  is a smooth, positive function tending to one at infinity. Assume that  $g$  is asymptotically flat, has nonnegative scalar curvature, and  $\partial M = \partial\Omega$  has zero mean curvature in  $(M, g)$ . Then:

$$m_{ADM}(g) \geq \left(\frac{A_1}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} + \dots + \left(\frac{A_k}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} + \frac{1}{2(n-1)\omega_{n-1}} \int_M R_g u^{-1} dV_g, \quad (10)$$

where, as before,  $A_1, \dots, A_k$  are the Euclidean areas of the components of  $\partial\Omega$ .

In other words, we merely assume that  $g$  is conformal to a scalar-flat metric that agrees with the Euclidean metric on  $\Sigma$  in a suitable sense.

*Proof.* The proof is nearly identical to that of Theorem 5, with one additional step. First, note the conformal transformation laws for the scalar curvature and mean curvature are the same as in the proof of Theorem 5, with  $\delta$  replaced by  $\bar{g}$ . (Here we are using the fact that  $\bar{g}$  is scalar-flat.) The only other issue in the extending the proof to this more general case is the following: the integral of  $\partial_{\bar{\nu}}(u)$  over a coordinate sphere at infinity measures the difference of the ADM masses of  $g$  and  $\bar{g}$ :

$$-\frac{2}{(n-2)\omega_{n-1}} \lim_{r \rightarrow \infty} \int_{S_r} \partial_{\bar{\nu}}(u) d\bar{A} = m_{ADM}(g) - m_{ADM}(M, \bar{g}), \quad (11)$$

where  $\bar{\nu}$  and  $d\bar{A}$  are the unit normal and area form with respect to  $\bar{g}$ . (Previously, in the case of Theorem 5,  $m_{ADM}(M, \bar{g}) = m_{ADM}(M, \delta) = 0$ .)

To recycle the proof of Theorem 5, we need only show that  $m_{ADM}(M, \bar{g}) \geq 0$ . Define a metric  $\tilde{g}$  on  $\mathbb{R}^n$  by gluing the metric  $\bar{g}$  on  $\mathbb{R}^n \setminus \Omega$  and the metric  $\delta$  defined on  $\Omega$ . By construction,  $\tilde{g}$  is Lipschitz across  $\Sigma$  and is smooth and scalar-flat away from  $\partial\Omega$ . Moreover, the mean curvatures of both sides of  $\partial\Omega$  agree by hypothesis. By Shi and Tam's generalization of the Witten's spin version positive mass theorem (Theorem 3.1 of [23]), the ADM mass of  $(M, \tilde{g})$  is nonnegative (this uses the fact that  $\mathbb{R}^n$  is a spin manifold). The proof is complete, because  $\tilde{g}$  and  $\bar{g}$  have the same ADM mass.  $\square$

*Remarks.* Interestingly, Theorem 7 does not follow from the Riemannian Penrose inequality in dimensions  $3 \leq n \leq 7$ , even if we decrease the right-hand side of (10) to  $\left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}$ . The RPI estimates  $m_{ADM}(g)$  from below in terms of any area-outer-minimizing minimal surface  $S$  in  $(M, g)$ . Certainly  $|S|_g \geq |S|_{\bar{g}}$  (since  $u \geq 1$ ), but  $|S|_{\bar{g}}$  need not be bounded below by  $|\Sigma|_{\bar{g}} = A$  (i.e.,  $\Sigma$  need not be area-outer-minimizing in  $(M, \bar{g})$ ).

**3.2. Removing the hypothesis on scalar curvature.** In Theorems 5 and 7, the non-negativity of scalar curvature implied superharmonicity of the conformal factor  $u$ , which in turn implied  $u \geq 1$ . Below we describe how to draw similar conclusions without assuming nonnegative scalar curvature.

*Theorem 8.* Suppose  $(M, g)$  is as in Theorem 7, except remove the assumption that  $g$  has nonnegative scalar curvature. Then

$$m_{ADM}(g) \geq \left(\frac{A_1}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} + \dots + \left(\frac{A_k}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} + \frac{1}{2(n-1)\omega_{n-1}} \int_M R_g u^{-2} dV_g. \quad (12)$$

Note the weighting factor on the integral of scalar curvature is  $u^{-2}$  rather than  $u^{-1}$  as before, and this term may have any sign.

Penrose inequalities that include a weighted integral of scalar curvature may be of interest in proving the conjectured “general” Penrose inequality for slices of spacetimes that are not totally geodesic [8].

*Proof.* Let  $\bar{g}, u$ , and  $g$  be as in the statement of Theorem 7, except without the assumption that  $g$  has nonnegative scalar curvature. Since  $u \rightarrow 1$  at infinity and  $\partial M$  is compact, we know that  $u$  is bounded above and below by positive constants. Using the definition of asymptotic flatness to obtain appropriate decay on the derivatives of  $u$ , we have that the functions  $\frac{\bar{\Delta}u}{u}$  and  $\frac{|\bar{\nabla}u|_{\bar{g}}^2}{u^2}$  are integrable on  $(M, \bar{g})$ , so

$$f := \overline{\operatorname{div}} \left( \frac{\bar{\nabla}u}{u} \right) = \frac{\bar{\Delta}u}{u} - \frac{|\bar{\nabla}u|_{\bar{g}}^2}{u^2}$$

is integrable, where  $\overline{\operatorname{div}}$  and  $\bar{\nabla}$  are the  $\bar{g}$ -divergence and gradient. Integrating by parts yields

$$\lim_{r \rightarrow \infty} \int_{S_r} \frac{\partial_{\bar{\nu}}(u)}{u} \bar{dA} - \int_{\partial M} \frac{\partial_{\bar{\nu}}(u)}{u} \bar{dA} = \int_M \frac{\bar{\Delta}u}{u} \bar{dV} - \int_M \frac{|\bar{\nabla}u|_{\bar{g}}^2}{u^2} \bar{dV}$$

where  $S_r$  is the coordinate sphere of radius  $r$ , and  $\bar{dV}$ ,  $\bar{dA}$  and  $\bar{\nu}$  are the volume form, area form, and unit normal with respect to  $\bar{g}$ . In the first term above, the  $u$  in the denominator may be ignored, since  $u \rightarrow 1$  at infinity. Multiplying by the constant  $-\frac{2}{(n-2)\omega_{n-1}}$ , applying the conformal transformation laws for the ADM mass (11), mean curvature (7), scalar curvature (5), and volume form, and discarding the  $|\bar{\nabla}u|^2$  term yields the inequality:

$$m_{ADM}(g) - m_{ADM}(\bar{g}) \geq \frac{1}{(n-1)\omega_{n-1}} \int_{\Sigma} \bar{H} \bar{dA} + \frac{1}{2(n-1)\omega_{n-1}} \int_M R_g u^{-2} dV_g,$$

where  $\bar{H}$  is the mean curvature of  $\Sigma$  with respect to  $\bar{g}$ . The hypothesis on  $\bar{g}$  allows us to replace  $\bar{H} \bar{dA}$  with  $H dA$ . To complete the proof, apply the Minkowski inequality to  $\int_{\Sigma} H dA$  as in the proof of Theorem 5, and use the nonnegativity of the ADM mass of  $\bar{g}$  as in the proof of Theorem 7.  $\square$

#### 4. INEQUALITIES FOR ZERO AREA SINGULARITIES

We recall the idea of a *zero area singularity*, which is in a sense dual to the idea of a black hole as a minimal surface. Suppose  $M$  is a smooth manifold with smooth compact boundary  $\partial M$ . Let  $g$  be a smooth Riemannian metric defined on the interior  $M \setminus \partial M$ . A connected component  $S$  of  $\partial M$  is said to be a *zero area singularity* (ZAS) of  $g$  if for all sequences  $\{S_i\}$  of hypersurfaces in  $M \setminus \partial M$  converging in the  $C^1$  sense to  $S$ , we have

$$\lim_{i \rightarrow \infty} |S_i|_g = 0,$$

where  $|S_i|_g$  is the hypersurface area of  $S_i$  with respect to  $g$ . (Note that  $C^1$  convergence depends only on the smooth structure of  $M$ .) The study of ZAS was initiated by Bray [3]. For more details and precise definitions, see [7] and the survey paper [4].

The motivating example of a manifold containing a zero area singularity is the Schwarzschild manifold of negative mass, described as follows. For  $n \geq 3$  and a real parameter



$m < 0$ , suppose  $M$  is  $\mathbb{R}^n$  minus the open ball about the origin of radius  $\left(\frac{|m|}{2}\right)^{\frac{1}{n-2}}$ . Let  $g_m$  be the metric on  $M$  defined by

$$g_m = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \delta.$$

It is not difficult to see that the boundary sphere  $\partial M$  is a ZAS of the metric  $g_m$ , since the conformal factor vanishes on  $\partial M$ .

In the context of this paper, it is natural to restrict to the class of *regular* ZAS, which are singularities that arise from deforming a smooth metric by a conformal factor that vanishes on a boundary component. More precisely, a metric  $g$  has a *regular ZAS* on a boundary component  $\Sigma$  if, on a neighborhood  $U$  of  $\Sigma$ , we have the equality  $g = u^{\frac{4}{n-2}} \bar{g}$ , where  $\bar{g}$  is a smooth Riemannian metric on  $U$  (up to and including on  $\Sigma$ ) and  $u \geq 0$  is a smooth function on  $U$  vanishing only on  $\Sigma$  and satisfying  $\partial_{\bar{\nu}} u > 0$  on  $\Sigma$ . (Here  $\bar{\nu}$  is the unit normal to  $\Sigma$  with respect to  $\bar{g}$ , pointing into the manifold.) For instance, the singularity in the Schwarzschild manifold of negative mass arises from such a construction (where  $\bar{g} = \delta$ ).

Analogous to the definition of the mass of a collection of black holes of total area  $A$  to be  $m_{BH} = \frac{1}{2} \left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}$ , Bray proposed the following formula for the “mass” of a collection of regular ZAS  $\Sigma$  of a metric  $g = u^{\frac{4}{n-2}} \bar{g}$ :

$$m_{ZAS}(\Sigma) = -\frac{2}{(n-2)^2} \left( \frac{1}{\omega_{n-1}} \int_{\Sigma} (\partial_{\bar{\nu}} u)^{\frac{2(n-1)}{n}} \bar{dA} \right)^{\frac{n}{n-1}},$$

where  $\bar{dA}$  is the area measure on  $\Sigma$  with respect to  $\bar{g}$  [3, 4, 7]. This negative real number depends only on the local geometry of  $g$  near  $\Sigma$  (not on the pair  $(\bar{g}, u)$ ) and produces the value  $m$  for the ZAS in the Schwarzschild manifold of mass  $m < 0$ .

Motivated by the Riemannian Penrose inequality, Bray conjectured that in an asymptotically flat manifold  $(M, g)$  of nonnegative scalar curvature for which every component of the compact boundary  $\Sigma = \partial M$  is a ZAS of  $g$ , the ADM mass ought to be bounded below in terms of the ZAS mass:

$$m_{ADM}(g) \geq m_{ZAS}(\Sigma) \quad (\text{conjectured}). \quad (13)$$

This inequality remains a conjecture. Some special cases in which (13) is known are:

- $n = 3$  and  $\Sigma$  is connected (due to Robbins [16], using inverse mean curvature flow),
- $3 \leq n \leq 7$  and  $g = u^{\frac{4}{n-2}} \bar{g}$ , where  $(M, \bar{g})$  satisfies the hypothesis of the Riemannian Penrose inequality (Theorem 2); see [3, 4, 7], or
- $(M, g)$  is a graph over  $\mathbb{R}^n$  in Minkowski space  $\mathbb{R}^{n,1}$ , and each of the ZAS is a level set of the graph function [14]. (But note that Lam’s definition of  $m_{ZAS}$  is somewhat different.)

We emphasize that the positive mass theorem does not apply to manifolds that contain regular ZAS; such manifolds are incomplete and may have negative ADM mass. Therefore (13) may be viewed as a generalization of the PMT, providing a lower bound for the ADM mass of manifolds that contain singularities.

*Setup:* Our goal is to prove a weakened version of (13) in the conformally flat case. For  $n \geq 3$ , suppose that  $\Omega \subset \mathbb{R}^n$  is a smooth bounded open set, such that  $M = \mathbb{R}^n \setminus \Omega$  is connected. Let  $\Sigma = \partial M$ . Assume  $u$  is a function on  $M$  with the following properties:

- (1)  $u \rightarrow 1$  at infinity and  $g = u^{\frac{4}{n-2}} \delta$  is asymptotically flat,
- (2)  $u$  vanishes precisely on  $\Sigma$ , and
- (3)  $\Delta u \leq 0$  (equivalently,  $g$  has nonnegative scalar curvature).

By the maximum principle,  $u > 0$  in the interior of  $M$  and  $\partial_\nu(u) > 0$  on  $\Sigma$ . Therefore, each component of  $\Sigma$  is a regular zero area singularity of  $g$ .

We first give a result that estimates the ADM mass  $m$  of  $(M, g)$  from below in terms of the ZAS mass and the Euclidean area  $A$  of  $\Sigma$ .

*Lemma 9.* *With the above setup,*

$$m_{ADM}(g) \geq m_{ZAS}(\Sigma) - \frac{1}{2} \left( \frac{A}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}.$$

*Proof.* Integrating  $\Delta u \leq 0$  over  $M$  gives the inequality (c.f. the proof of Theorem 5):

$$m \geq -\frac{2}{(n-2)\omega_{n-1}} \int_{\partial M} \partial_\nu(u) dA,$$

where  $m = m_{ADM}(g)$ . Apply Hölder's inequality, then add and subtract  $\frac{1}{2} \left( \frac{A}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}$  to obtain:

$$\begin{aligned} m &\geq -\frac{2}{(n-2)\omega_{n-1}} \left( \int_{\Sigma} (\partial_\nu u)^{\frac{2(n-1)}{n}} dA \right)^{\frac{n}{2(n-1)}} A^{\frac{n-2}{2(n-1)}} \\ &= \underbrace{\frac{1}{2} \left( \frac{A}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} - \frac{2}{(n-2)} \left( \frac{1}{\omega_{n-1}} \int_{\Sigma} (\partial_\nu u)^{\frac{2(n-1)}{n}} dA \right)^{\frac{n}{2(n-1)}} \left( \frac{A}{\omega_{n-1}} \right)^{\frac{n-2}{2(n-1)}}}_{\text{underbraced terms}} - \frac{1}{2} \left( \frac{A}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}. \end{aligned}$$

At this point, we can invoke an argument of Bray [3, 7], viewing the underbraced terms as a degree-two polynomial in the variable  $x = \left( \frac{A}{\omega_{n-1}} \right)^{\frac{n-2}{2(n-1)}}$ . Minimizing over all  $x \in \mathbb{R}$  gives

$$m \geq -\frac{2}{(n-2)^2} \left( \frac{1}{\omega_{n-1}} \int_{\Sigma} (\partial_\nu u)^{\frac{2(n-1)}{n}} dA \right)^{\frac{n}{n-1}} - \frac{1}{2} \left( \frac{A}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}.$$

Recognizing the first term on the right-hand side as  $m_{ZAS}(\Sigma)$  completes the proof.  $\square$

The estimate in Lemma 9 is unsatisfactory for the reason that the error term is additive rather than multiplicative. The following theorem provides a remedy.

*Theorem 10.* *With the above setup,*

$$m_{ADM}(g) \geq m_{ZAS}(\Sigma) \left( 1 + \frac{1}{4} \iota^2 \right),$$

where

$$\iota = \left( \frac{A}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} \bigg/ \left( \frac{V}{\beta_n} \right)^{\frac{n-2}{n}}$$

is the isoperimetric ratio of  $\Omega$ . Here,  $A$  and  $V$  are the Euclidean area and volume of  $\partial\Omega$  and  $\Omega$ .

The classical isoperimetric inequality is the statement  $\iota \geq 1$  [15]. Since  $m_{ZAS}(\Sigma) < 0$ , the estimate of Theorem 10 is weaker than the conjectured inequality (13). In the model case in which  $\Omega$  is a round ball and  $u$  is harmonic (or equivalently,  $(M, g)$  is a negative-mass Schwarzschild manifold), the inequality is suboptimal by a factor of  $\frac{5}{4}$ .

The idea of the proof is to use Lemma 9 in conjunction with an upper bound on the area  $A$  in terms of the isoperimetric ratio of  $\Omega$  and the absolute value of the ZAS mass. Before proceeding, recall that the *capacity* of the bounded open set  $\Omega \subset \mathbb{R}^n$  is the number:

$$\text{cap}(\Omega) = \frac{1}{(n-2)\omega_{n-1}} \inf_{\psi} \left\{ \int_{\mathbb{R}^n \setminus \Omega} |\nabla \psi|^2 dV \right\},$$

where the infimum is taken over all locally-Lipschitz functions  $\psi$  on  $\mathbb{R}^n \setminus \Omega$  that vanish on  $\partial\Omega$  and approach one at infinity. (The geometric quantities in this expression are taken with respect to the Euclidean metric.) A standard exercise shows that the infimum is attained by the unique harmonic function  $\varphi$  that vanishes on  $\Sigma = \partial\Omega$  and approaches one at infinity. Applying Stokes' theorem twice and invoking the boundary conditions of  $\varphi$  lead to the equivalent expression for the capacity:

$$\text{cap}(\Omega) = \frac{1}{(n-2)\omega_{n-1}} \int_{\Sigma} \partial_{\nu} \varphi dA.$$

By this convention, a unit ball in  $\mathbb{R}^n$  has unit capacity.

*Proof of Theorem 10.* Let  $\varphi$  be as above. By the maximum principle and the superharmonicity of  $u$ , we have  $\partial_{\nu} \varphi \leq \partial_{\nu} u$  on  $\Sigma$ , so

$$\text{cap}(\Omega) \leq \frac{1}{(n-2)\omega_{n-1}} \int_{\Sigma} \partial_{\nu} u dA.$$

Apply Hölder's inequality to the right-hand side to obtain:

$$\text{cap}(\Omega) \leq \frac{1}{(n-2)\omega_{n-1}} \left( \int_{\Sigma} (\partial_{\nu} u)^{\frac{2(n-1)}{n}} dA \right)^{\frac{n}{2(n-1)}} A^{\frac{n-2}{2(n-1)}}$$

The inequality of Poincaré–Faber–Szegő [15] (c.f. appendix A herein) relating the capacity of a region to its volume states:

$$\text{cap}(\Omega) \geq \left( \frac{V}{\beta_n} \right)^{\frac{n-2}{n}}.$$

Combining the last two inequalities, squaring, and rearranging, we have:

$$\left( \frac{V}{\beta_n} \right)^{\frac{2(n-2)}{n}} \leq \frac{1}{(n-2)^2} \left( \frac{1}{\omega_{n-1}} \int_{\Sigma} (\partial_{\nu} u)^{\frac{2(n-1)}{n}} dA \right)^{\frac{n}{n-1}} \left( \frac{A}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}$$

Using the definition of ZAS mass and isoperimetric constant, we have:

$$\left( \frac{A}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} \leq \frac{\iota^2}{2} |m_{ZAS}(\Sigma)|. \quad (14)$$

Combining (14) with the estimate from Lemma 9 establishes the result.  $\square$

*Remarks.* Theorem 10 does not immediately generalize to background metrics other than  $\delta$  in the sense of Theorem 7, for the reason that the capacity–volume inequality depends strongly on the global Euclidean nature of the region outside  $\Omega$ . On the other hand, Theorem 10 requires no topological or convexity assumptions on  $\Omega$ .

## 5. A MIXED INEQUALITY FOR BLACK HOLES AND ZAS

Based on considerations in classical physics pertaining to potential energy, Bray conjectured that in an asymptotically flat manifold (of nonnegative scalar curvature) with boundary  $\Sigma$  consisting of area-outer-minimizing minimal surfaces  $\Sigma_+$  (of total area  $|\Sigma_+|_g$ ) and zero area singularities  $\Sigma_-$ , the ADM mass ought to be bounded below as follows [3, 7]:

$$m_{ADM}(g) \geq \frac{1}{2} \left( \frac{|\Sigma_+|_g}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} + m_{ZAS}(\Sigma_-) \quad (\text{conjectured}) \quad (15)$$

Consider this problem in the conformally flat case, with the following setup. Suppose that  $\Omega_+$  and  $\Omega_-$  are smooth bounded open sets in  $\mathbb{R}^n$  whose closures do not intersect. Assume that  $\Omega_+$  is mean-convex and every connected component is star-shaped. Set  $\Omega = \Omega_+ \dot{\cup} \Omega_-$ , and assume  $M = \mathbb{R}^n \setminus \Omega$  is connected. Let  $\Sigma_{\pm}$  be the connected components of  $\partial\Omega_{\pm}$  in  $M$ , so  $\partial M = \Sigma_+ \dot{\cup} \Sigma_-$ . Let  $u \geq 0$  be a smooth function on  $M$  with the following properties:

- (1)  $u$  vanishes precisely on  $\Sigma_-$ ,
- (2)  $u \rightarrow 1$  at infinity and  $g = u^{\frac{4}{n-2}}\delta$  is asymptotically flat,
- (3)  $\Delta u \leq 0$  (equivalently,  $g$  has nonnegative scalar curvature away from  $\Sigma_-$ ), and
- (4)  $\Sigma_+$  has zero mean curvature with respect to  $g$ . Equivalently,

$$Hu + \frac{2(n-1)}{n-2} \partial_{\nu} u = 0.$$

In the metric  $g$ , each component of  $\Sigma_+$  is a minimal surface, and each component of  $\Sigma_-$  is a regular ZAS. (Note  $\partial_{\nu} u > 0$  on  $\Sigma_-$  by the maximum principle.)

We make the *ad hoc* assumption that  $u \geq 1$  on  $\Sigma_+$  and proceed as follows. First, integrate  $\Delta u \leq 0$  over  $M$ . Arguments from Theorem 5 and Lemma 9 give the inequalities:

$$\begin{aligned} m_{ADM}(g) &\geq \frac{1}{(n-1)\omega_{n-1}} \int_{\Sigma_+} H u dA - \frac{2}{(n-2)\omega_{n-1}} \int_{\Sigma_-} \nu(u) dA \\ &\geq \left( \frac{A_+}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} + m_{ZAS}(\Sigma_-) - \frac{1}{2} \left( \frac{A_-}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}, \end{aligned}$$

where  $A_{\pm}$  is the Euclidean area of  $\Sigma_{\pm}$ . We essentially apply the same steps as in the proof of Theorem 10. Let  $\varphi$  be the  $\delta$ -harmonic function on  $M \setminus \Omega_-$  that vanishes on  $\Sigma_-$  and tends to one at infinity. Since  $u = 0$  on  $\Sigma_-$ ,  $u \geq 1$  on  $\Sigma_+$  by assumption, and  $u$  is superharmonic, we have  $\partial_{\nu} \varphi \leq \partial_{\nu} u$  on  $\Sigma_-$ . Running through the same argument as in Theorem 10 leads to:

$$m \geq \left( \frac{A_+}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} + m_{ZAS}(\Sigma_-) \left( 1 + \frac{1}{4} \iota_-^2 \right),$$

where  $\iota_-$  is the isoperimetric constant of  $\Omega_-$ . This is a weakened version of the conjectured inequality (15), assuming  $u \geq 1$  on  $\Sigma_+$ .

An interesting open problem in elliptic PDE is to determine whether  $u \geq 1$  on  $\Sigma_+$  is automatic for a superharmonic function  $u$  with the above boundary conditions.

## 6. CONCLUDING REMARKS

In the Riemannian Penrose inequality (Theorem 2), it is well-known that the hypothesis that the boundary  $\Sigma$  be area-outer-minimizing is crucial. Indeed, one may easily construct examples of rotationally-symmetric, asymptotically flat manifolds  $(M, g)$  of nonnegative scalar curvature and minimal boundary  $\Sigma$  of area  $A$  such that the ratio  $m_{ADM}(g)/A^{\frac{n-2}{n-1}}$  is arbitrarily small. In this case,  $\Sigma$  is “hidden” behind some area-outer-minimizing minimal surface  $\tilde{\Sigma}$  of area  $\tilde{A}$ , and the Riemannian Penrose inequality holds for the region exterior to  $\tilde{\Sigma}$  (see figure 1 of [12]).

The theorems of this paper do not require the boundary to be area-outer-minimizing, which is perhaps philosophically the reason why we do not recover a sharp version of the RPI for conformally flat manifolds. This phenomenon (together with the difficulty of utilizing the area-outer-minimizing hypothesis) was pointed out by Bray and Iga [6]. Nevertheless, rotationally-symmetric manifolds are conformally flat, so the above examples of hidden minimal surfaces make it surprising that we can prove any inequality at all.

## APPENDIX A. THE POINCARÉ–FABER–SZEGÖ CAPACITY–VOLUME INEQUALITY

For reference, we include a proof of the capacity–volume inequality in dimension  $n \geq 3$ , based entirely on the  $n = 3$  case of [15]. Recall the definition of capacity from section 4.

*Theorem 11 (Poincaré–Faber–Szegő).* *Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$  with smooth boundary such that  $\mathbb{R}^n \setminus \Omega$  is connected. Then*

$$\text{cap}(\Omega) \geq \left( \frac{V}{\beta_n} \right)^{\frac{n-2}{n}}, \quad (16)$$

where  $\text{cap}(\Omega)$  and  $V$  are the capacity and volume of  $\Omega$ , respectively.

*Proof.* Let  $\varphi$  be the unique function on  $\mathbb{R}^n \setminus \Omega$  that vanishes on  $\partial\Omega$ , is harmonic on  $\mathbb{R}^n \setminus \Omega$ , and approaches 1 at infinity. Then

$$(n-2)\omega_{n-1} \text{cap}(\Omega) = \int_{\mathbb{R}^n \setminus \Omega} |\nabla \varphi|^2 dV.$$

For  $t \in [0, 1]$ , let  $\Sigma_t$  be the level set  $\varphi^{-1}(t)$  (smooth for almost every  $t$ ). By the co-area formula,

$$\int_{\mathbb{R}^n \setminus \Omega} |\nabla \varphi|^2 dV = \int_0^1 \int_{\Sigma_t} |\nabla \varphi|^2 \frac{1}{|\nabla \varphi|} dA_t dt,$$

where  $dA_t$  is the area form on  $\Sigma_t$ . By the Schwarz inequality,

$$|\Sigma_t|^2 \leq \left( \int_{\Sigma_t} |\nabla \varphi| dA_t \right) \left( \int_{\Sigma_t} \frac{1}{|\nabla \varphi|} dA_t \right),$$

where  $|\Sigma_t|$  is the area of  $\Sigma_t$ . Combining the last equality and inequality produces:

$$\int_{\mathbb{R}^n \setminus \Omega} |\nabla \varphi|^2 dV \geq \int_0^1 \frac{|\Sigma_t|^2}{\int_{\Sigma_t} \frac{1}{|\nabla \varphi|} dA_t} dt.$$

Let  $V(t)$  be the volume of the region bounded by  $\Sigma_t$ , so again by the co-area formula:

$$V(t) = \text{vol}(\Omega) + \int_0^t \int_{\Sigma_t} \frac{1}{|\nabla \varphi|} dA_t dt,$$

and therefore

$$V'(t) = \int_{\Sigma_t} \frac{1}{|\nabla \varphi|} dA_t.$$

Combining the above gives

$$\begin{aligned} (n-2)\omega_{n-1} \text{cap}(\Omega) &\geq \int_0^1 \frac{|\Sigma_t|^2}{V'(t)} dt \\ &\geq \int_0^1 \frac{\omega_{n-1}^2 \left(\frac{V(t)}{\beta_n}\right)^{\frac{2(n-1)}{n}}}{V'(t)} dt, \end{aligned}$$

where we have used the isoperimetric inequality on the second line. Let  $R(t)$  be the radius of the sphere that has volume equal to  $V(t)$ , i.e.,  $V(t) = \beta_n R(t)^n$ . Then  $V'(t) = n\beta_n R(t)^{n-1} R'(t)$ , so

$$(n-2)\omega_{n-1} \text{cap}(\Omega) \geq \int_0^1 \frac{\omega_{n-1} R(t)^{n-1}}{R'(t)} dt, \quad (17)$$

having used the fact  $n\beta_n = \omega_{n-1}$ .

Now, let  $\tilde{\Omega}$  be the open ball about the origin with the same volume as  $\Omega$ . Let  $\tilde{\Sigma}_t$  be the sphere about the origin of radius  $R(t)$ , with area form  $d\tilde{A}_t$ . Let  $\tilde{\varphi}$  be the function that equals  $t$  on  $\tilde{\Sigma}_t$ . We continue inequality (17), using the fact that  $\omega_{n-1} R(t)^{n-1} = \int_{\tilde{\Sigma}_t} d\tilde{A}_t$  and the observation that  $|\nabla \tilde{\varphi}| = \frac{1}{R'(t)}$  on  $\tilde{\Sigma}_t$ :

$$\begin{aligned} (n-2)\omega_{n-1} \text{cap}(\Omega) &\geq \int_0^1 \int_{\tilde{\Sigma}_t} |\nabla \tilde{\varphi}| d\tilde{A}_t dt && \text{(by (17))} \\ &\geq \int_{\mathbb{R}^n \setminus \Omega^*} |\nabla \tilde{\varphi}|^2 dV && \text{(co-area formula)} \\ &\geq (n-2)\omega_{n-1} \text{cap}(\tilde{\Omega}), \end{aligned}$$

where on the last line we used the definition of the capacity of  $\tilde{\Omega}$  and the fact that  $\tilde{\varphi}$  is locally-Lipschitz with appropriate boundary conditions. It is easy to check that equality in (16) holds for round balls. Since  $\Omega$  has the same volume as  $\tilde{\Omega}$ , the proof is complete.  $\square$

## REFERENCES

- [1] R. Arnowitt, S. Deser, and C. W. Misner, *Coordinate invariance and energy expressions in general relativity*, Phys. Rev. (2) **122** (1961), 997–1006.
- [2] R. Bartnik, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math. **39** (1986), no. 5, 661–693.
- [3] H.L. Bray, *Negative point mass singularities in general relativity*, Global problems in mathematical relativity (Isaac Newton Institute, University of Cambridge, August 30, 2005), available at <http://www.newton.ac.uk/webseminars/pg+ws/2005/gmr/0830/bray/>.

- [4] ———, *On the positive mass, Penrose, and ZAS inequalities in general dimension* (H.L. Bray and W.P. Minicozzi, eds.), *Surveys in Geometric Analysis and Relativity in Honor of Richard Schoen's 60th Birthday*, Higher Education Press/International Press, Beijing/Boston, 2010, accepted.
- [5] ———, *Proof of the Riemannian Penrose inequality using the positive mass theorem*, *J. Differential Geom.* **59** (2001), no. 2, 177–267.
- [6] H.L. Bray and K. Iga, *Superharmonic functions in  $\mathbf{R}^n$  and the Penrose inequality in general relativity*, *Comm. Anal. Geom.* **10** (2002), no. 5, 999–1016.
- [7] H.L. Bray and J.L. Jauregui, *A geometric theory of zero area singularities in general relativity* (2009), available at <http://arxiv.org/abs/0909.0522>.
- [8] H.L. Bray and M.A. Khuri, *P.D.E.'s which imply the Penrose conjecture*, *Asian J. Math.* (2011, accepted).
- [9] H.L. Bray and D.A. Lee, *On the Riemannian Penrose inequality in dimensions less than eight*, *Duke Math. J.* **148** (2009), no. 1, 81–106.
- [10] G.W. Gibbons, *Collapsing shells and the isoperimetric inequality for black holes*, *Classical Quantum Gravity* **14** (1997), no. 10, 2905–2915.
- [11] P. Guan and J. Li, *The quermassintegral inequalities for  $k$ -convex starshaped domains*, *Adv. Math.* **221** (2009), no. 5, 1725–1732.
- [12] G. Huisken and T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, *J. Differential Geom.* **59** (2001), no. 3, 353–437.
- [13] M.-K. G. Lam, *The graph cases of the Riemannian positive mass and Penrose inequalities in all dimensions* (2010), available at <http://arxiv.org/abs/1010.4256>.
- [14] ———, *The graph cases of the Riemannian positive mass and Penrose inequalities in all dimensions*, Ph.D. thesis, Duke University, 2011.
- [15] G. Pólya and G. Szegő, *Isoperimetric Inequalities in Mathematical Physics*, *Annals of Mathematics Studies*, no. 27, Princeton University Press, Princeton, N. J., 1951.
- [16] N. Robbins, *Zero area singularities in general relativity and inverse mean curvature flow*, *Classical Quantum Gravity* **27** (2010), no. 2.
- [17] R. Schneider, *Convex bodies: the Brunn–Minkowski theory*, *Encyclopedia of Mathematics and its Applications*, vol. 44, Cambridge University Press, 1993.
- [18] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, *J. Differential Geom.* **20** (1984), no. 2, 479–495.
- [19] ———, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics*, *Topics in calculus of variations* (Montecatini Terme, 1987), *Lecture Notes in Math.*, vol. 1365, Springer, Berlin, 1989, pp. 120–154.
- [20] R. Schoen and S.-T. Yau, *On the proof of the positive mass conjecture in general relativity*, *Commun. Math. Phys.* **65** (1979), 45–76.
- [21] F. Schwartz, *A volumetric Penrose inequality for conformally flat manifolds*, *Ann. Henri Poincaré* **12** (2011), 67–76.
- [22] A. Freire and F. Schwartz, *Mass-capacity inequalities for conformally flat manifolds with boundary* (2011), available at <http://arxiv.org/abs/1107.1407>.
- [23] Y. Shi and L.-F. Tam, *Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature*, *J. Differential Geom.* **62** (2002), no. 1, 79–125.
- [24] E. Witten, *A new proof of the Positive Energy Theorem*, *Commun. Math. Phys.* **80** (1981), 381–402.

DEPARTMENT OF MATHEMATICS, DAVID RITTENHOUSE LAB, 209 SOUTH 33RD STREET, PHILADELPHIA, PA 19104,

*E-mail address:* [jauregui@math.upenn.edu](mailto:jauregui@math.upenn.edu)